7.1 Let $(\mathcal{M}, g)$ be a Riemannian manifold and $p \in \mathcal{M}$. Let $(x^1, \dots, x^n)$ be normal coordinates centered at p. Show that the components of the metric g satisfy at $p = (0, \dots, 0)$ the following cyclic identity for any $i, j, k, l \in \{1, \dots, n\}$:

$$\partial_i \partial_j g_{kl}|_p + \partial_j \partial_k g_{il}|_p + \partial_k \partial_i g_{jl}|_p = 0.$$

(Hint: recall that the Gauss lemma is equivalent to the statement that, in normal coordinates, $g_{ij} x^j = \delta_{ij} x^j$. Differentiate this relation a few times and evaluate at $(0, \dots, 0)$.)

Manipulating the above formula, show that

$$\partial_i \partial_j g_{kl}|_p = \partial_k \partial_l g_{ij}|_p.$$

7.2 In this exercise, we will compute the expression in polar coordinates of the three model geometries in 2 dimensions (this was originally part of the last exercise last week).

(a) As a warm up, express in polar coordinates centered at the origin the flat metric g_E on $\mathbb{R}^2$.

(b) Let $(\mathbb{H}^2, g_{\mathbb{H}})$ be the hyperbolic plane (see Exercise 6.4 for an expression of the metric in the Poincare disc model, when $\mathbb{H}^2$ is identified with the interior of the unit disc). Let p be a point in the hyperbolic plane. Compute the metric in polar coordinates around p. *(Hint: Working in the Poincaré disc model, it suffices to only consider the case when p is at the origin, since any point $p \in \mathbb{D}^2$ can be mapped to any other point in $\mathbb{D}^2$ via an isometry. What are the geodesics in $(\mathbb{D}^2, g_{\mathbb{D}})$ emanating from the origin?)*

(c) How is the round metric $(\mathbb{S}^2, g_{\mathbb{S}^2})$ expressed in polar coordinates around a point $p \in \mathbb{S}^2$?

(d) In all of the three Riemannian surfaces (S, g) considered above, compute the volume of the metric ball of radius $r > 0$ centered at any point $p \in (S, g)$ (due to the symmetry of the above spaces, the precise choice of p is irrelevant). Denoting with $B_{(S,g)}[r]$ the corresponding ball, show that

$$\mathrm{Vol}(B_{(\mathbb{S}^2, g_{\mathbb{S}^2})}[r]) < \mathrm{Vol}(B_{(\mathbb{R}^2, g_E)}[r]) < \mathrm{Vol}(B_{(\mathbb{H}^2, g_{\mathbb{H}^2})}[r]).$$

7.3 *The Euler–Lagrange equations.* Let $\Omega \subset \mathbb{R}^n$ be an open domain and let $(t, x, v) \to \mathcal{L}(t, x, v) \in \mathbb{R}$ be a smooth function for $(t, x, v) \in [a, b] \times \Omega \times \mathbb{R}^n$. For any smooth map $f : [a, b] \to \Omega$, we will define it's action with respect to $\mathcal{L}$ by the relation

$$S_{\mathcal{L}}[f] \doteq \int_a^b \mathcal{L}(t, f(t), \frac{df}{dt}(t))\, dt.$$

Let $F : (-\delta, \delta) \times [a, b] \to \Omega$ be a smooth variation of f, i.e. a smooth function satisfying

$$F(0, \cdot) = f(\cdot).$$

Show that

$$\frac{d}{ds}S_{\mathcal{L}}[F(s,\cdot)]\big|_{s=0} = \big[\frac{\partial F^i}{\partial s}(0,t)\cdot\partial_{v^i}\mathcal{L}(t,f(t),f'(t))\big]_{t=a}^{b} + \int_a^b \frac{\partial F^i}{\partial s}(0,t)\Big(\partial_{x^i}\mathcal{L}(t,f(t),f'(t)) - \frac{d}{dt}\big(\partial_{v^i}\mathcal{L}(t,f(t),f'(t))\big)\Big)\,dt, \tag{1}$$

where $\partial_{x^i}\mathcal{L}$ and $\partial_{v^i}\mathcal{L}$ denote the corresponding partial derivative of the function

$$\mathcal{L} = \mathcal{L}(t; x^1,\dots,x^n; v^1,\dots,v^n)$$

with respect to the Cartesian coordinates x^i and v^i on Ω and $\mathbb{R}^n$, respectively (*Hint: After applying the* $\frac{\partial}{\partial s}$ *derivative inside the integral defining* $S_{\mathcal{L}}$*, perform an integration by parts on the term* $\partial_s\partial_t F(s,t)$).

Deduce that if $f:[a,b]\to\Omega$ is a stationary point of $S_{\mathcal{L}}$ under *all* variations that fix the endpoints $t=a,b$ (i.e. $F(s,a)=f(a)$ and $F(s,b)=f(b)$), then f satisfies the *Euler–Lagrange equations*:

$$\partial_{x^i}\mathcal{L}(t,f(t),f'(t)) - \frac{d}{dt}\big(\partial_{v^i}\mathcal{L}(t,f(t),f'(t))\big) = 0.$$

Remark. In classical mechanics, $f:[a,b]\to\Omega$ can be thought of as the trajectory of a particle moving in the domain Ω for time $t\in[a,b]$. In this case, we can define $\mathcal{L}$ to be the *Lagrangian* of the particle; in the case when the particle moves under the influence of a conservative force (i.e. one which can be written as minus the gradient of a potnetial), the Lagrangian takes the form of the difference between the kinetic and potential energy of the particle:

$$\mathcal{L}(t,x,v) = \frac{1}{2}mv^2 - U(x).$$

The functional $S_{\mathcal{L}}$ is called the *action* of the trajectory f. An equivalent way of formulating Newtonian mechanics is by assuming the *principle of least action*: The particle moves along a trajectory for which the action is stationary among all paths between $f(a)$ and $f(b)$. You can verify that, in the case of a conservative force, the Euler–Lagrange equations are the standard Newtonian equations of motion for the particle:

$$m\frac{d^2 f^i}{dt^2}(t) = -\partial_i U\circ f(t).$$

***7.4** *Geodesics as stationary points of the energy functional.* We will now extend the formalism of the previous exercise to the realm of manifolds. Let $(\mathcal{M},g)$ be a smooth Riemannian manifold. Let $\mathcal{L}: T\mathcal{M}\to\mathbb{R}$ be a smooth function; for any $x\in\mathcal{M}$ and $v\in T_x\mathcal{M}$, we will denote with $\mathcal{L}(x,v)$ the value of $\mathcal{L}$ at $(x,v)\in T\mathcal{M}$. For any smooth curve $\gamma:[a,b]\to\mathcal{M}$, we will define the action of γ with respect to $\mathcal{L}$ by

$$S_{\mathcal{L}}[\gamma] \doteq \int_a^b \mathcal{L}[\gamma(t),\dot\gamma(t)]\,dt.$$

(a) Let $\gamma : [a, b] \to \mathcal{M}$ be a given curve and $\phi : (-\delta, \delta) \times [a, b] \to \mathcal{M}$ be a smooth variation of γ which is entirely contained in a coordinate chart $(x^1, \ldots, x^n)$ on $\mathcal{M}$; we will denote with $\frac{\partial \phi_s}{\partial s}|_{s=0}$ the variation field along γ (as we did in class). Show that (1) also holds in this case, i.e.

$$\frac{d}{ds} S_{\mathcal{L}}[\phi_s]\big|_{s=0} = \Big[\partial_{v^i} \mathcal{L}(\gamma(t), \dot{\gamma}(t)) \cdot \frac{\partial \phi_s^i}{\partial s}|_{s=0}(t)\Big]_{t=a}^{b} + \int_a^b \Big(\partial_{x^i} \mathcal{L}(\gamma(t), \dot{\gamma}(t)) - \frac{d}{dt}\big(\partial_{v^i} \mathcal{L}(\gamma(t), \dot{\gamma}(t))\big)\Big) \cdot \frac{\partial \phi_s^i}{\partial s}|_{s=0}(t)\, dt,$$

where, in the local coordinates $(x^1, \ldots, x^n; v^1, \ldots, v^n)$ on $T\mathcal{M}$ associated to $(x^1, \ldots, x^n)$ (recall that $v^i(V) = dx^i(V)$ for any $V \in \Gamma(\mathcal{M})$), $\partial_{x^i} \mathcal{L}$ and $\partial_{v^i} \mathcal{L}$ denote the partial derivatives of $\mathcal{L}(x^1, \ldots, x^n; v^1, \ldots, v^n)$ with respect to the corresponding variables.

Moreover, if γ is a stationary point for $S_{\mathcal{L}}$ for all variations ϕ_s with $\phi_s(a) = \gamma(a)$ and $\phi_s(b) = \gamma(b)$, then

$$\partial_{x^i} \mathcal{L}(\gamma(t), \dot{\gamma}(t)) - \frac{d}{dt}\big(\partial_{v^i} \mathcal{L}(\gamma(t), \dot{\gamma}(t))\big) = 0, \quad i = 1, \ldots, n.$$

(b) Let us now examine the case when

$$\mathcal{L}(x, v) = \frac{1}{2} g|_x(v, v) \quad \text{for } x \in \mathcal{M}, v \in T_x\mathcal{M}$$

(this can be thought of as an extension of the Newtonian function for the kinetic energy in the setting of Riemannian manifolds). In this case, the action $S_{\mathcal{L}}$ is known as the *energy* functional (which we also saw in Ex. 2.1):

$$\mathcal{E}[\gamma] = \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))\, dt.$$

Show that if ϕ_s is a variation of a smooth curve $\gamma : [a, b] \to \mathcal{M}$, not necessarily contained in a single coordinate chart, then

$$\frac{d}{ds} \mathcal{E}[\phi_s]|_{s=0} = \big\langle \frac{\partial \phi_s}{\partial s}|_{s=0}, \dot{\gamma} \big\rangle_g \Big|_{t=a}^{b} - \int_a^b \big\langle \frac{\partial \phi_s}{\partial s}|_{s=0}, \nabla_{\dot{\gamma}} \dot{\gamma} \big\rangle_g\, dt.$$

This is known as the 1^{st} variation formula for the energy.(*Hint: Break up the variation into smaller intervals in t such that each one is contained inside a single coordinate chart.*) Deduce that if γ is a stationary point for the energy under all variations which fix the endpoints $\gamma(a)$, $\gamma(b)$, then γ is a geodesic.

Remark. In contrast to the case of stationary curves for the length functional (which are reparametrizations of geodesics, not necessarily with constant speed), a reparametrization of a geodesic is not a stationary point for $\mathcal{E}[\gamma]$. Thus, $\mathcal{E}[\gamma]$ can be used to single out "properly parametrized" geodesics via a minimization process.

(c) For any $p \in \mathcal{M}$, let $\sigma : [a,b] \to \Omega_p \subset T_p\mathcal{M}$ be a smooth curve (Ω_p is the domain of definition of the exponential map $\exp_p$). Show that

$$\frac{d}{ds}\big(\|\sigma(s)\|^2_{g|_p}\big) = \big\langle d\exp_p|_{\sigma(s)}\dot{\sigma}(s),\, d\exp_p|_{\sigma(s)}\sigma(s)\big\rangle_g,$$

where we view $\sigma(s)$ both as point in Ω_p and a vector in $T_{\sigma(s)}\Omega_p$ (namely as the tangent vector of the line $t \to \sigma(s)t$ at $t = 1$). *Hint: What is the energy of the geodesic $t \to \exp_p(\sigma(s)t)$, $t \in [0,1]$?* Deduce from the above formula the statement of the lemma of Gauss.